Cantor sets and cyclicity in weighted Dirichlet spaces

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CANTOR SETS AND CYCLICITY IN WEIGHTED DIRICHLET SPACES

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Abstract. We treat the problem of characterizing the cyclic vectors in the weighted Dirichlet spaces, extending some of our earlier results in the classical Dirichlet space. The absence of a Carleson-type formula for weighted Dirichlet integrals necessitates the introduction of new techniques.

1. Introduction

In this paper we study the weighted Dirichlet spaces $\mathcal{D}_\alpha$ $(0 \leq \alpha \leq 1)$, defined by

$$\mathcal{D}_\alpha := \Big\{ f \in \mathrm{hol}(\mathbb{D}) : \mathcal{D}_\alpha(f) := \frac{1}{\pi}\int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^\alpha \, dA(z) < \infty \Big\}.$$

Here $\mathbb{D}$ denotes the open unit disk, and dA is area measure on $\mathbb{D}$. Clearly $\mathcal{D}_\alpha$ is a Hilbert space with respect to the norm $\|\cdot\|_\alpha$ given by

$$\|f\|_\alpha^2 := |f(0)|^2 + \mathcal{D}_\alpha(f).$$

A classical calculation shows that, if $f(z) = \sum_{n\geq 0} a_n z^n$, then

$$\|f\|_\alpha^2 \asymp \sum_{n\geq 0} (n+1)^{1-\alpha}|a_n|^2.$$

Note that $\mathcal{D}_1 = H^2$ is the usual Hardy space, and $\mathcal{D}_0$ is the classical Dirichlet space (thus our labelling convention follows [1] rather than [2]).

An *invariant subspace* of $\mathcal{D}_\alpha$ is a closed subspace M of $\mathcal{D}_\alpha$ such that $zM \subset M$. Given $f \in \mathcal{D}_\alpha$, we denote by $[f]_{\mathcal{D}_\alpha}$ the smallest invariant subspace of $\mathcal{D}_\alpha$ containing f, namely the closure in $\mathcal{D}_\alpha$ of $\{pf : p \text{ a polynomial}\}$. We say that f is *cyclic* for $\mathcal{D}_\alpha$ if $[f]_{\mathcal{D}_\alpha} = \mathcal{D}_\alpha$. The survey article [8] gives a brief history of invariant subspaces and cyclic functions in the classical case $\alpha = 0$.

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Our goal is to characterize the cyclic functions of $\mathcal{D}_\alpha$. In order to state our results, we introduce the notion of α-capacity. For $\alpha \in [0,1)$, we define the kernel function $k_\alpha : \mathbb{R}^+ \to \mathbb{R} \cup \{\infty\}$ by

$$k_\alpha(t) := \begin{cases} 1/t^\alpha, & 0 < \alpha < 1, \\ \log(1/t), & \alpha = 0. \end{cases}$$

The *α-energy* of a (Borel) probability measure μ on $\mathbb{T}$ is defined by

$$I_\alpha(\mu) := \iint k_\alpha(|\zeta - \zeta'|)\, d\mu(\zeta)\, d\mu(\zeta').$$

A standard calculation gives

$$I_\alpha(\mu) \asymp \sum_{n \geq 0} \frac{|\widehat{\mu}(n)|^2}{(1+n)^{1-\alpha}}.$$

The *α-capacity* of a Borel subset E of $\mathbb{T}$ is defined by

$$C_\alpha(E) := 1/\inf\{I_\alpha(\mu) : \mathcal{P}(E)\},$$

where $\mathcal{P}(E)$ denotes the set of all probability measures supported on compact subsets of E. In particular, $C_\alpha(E) > 0$ if and only if there exists a probability measure μ supported on a compact subset of E and having finite α-energy. If $\alpha = 0$, then C_0 is the classical logarithmic capacity.

We recall a result due to Beurling and Salem–Zygmund [5, §V, Theorem 3] about radial limits of functions in the weighted Dirichlet spaces. If $f \in \mathcal{D}_\alpha$, then $f^*(\zeta) := \lim_{r \to 1-} f(r\zeta)$ exists for all $\zeta \in \mathbb{T}$ outside a set of α-capacity zero.

The following theorem gives two necessary conditions for cyclicity in $\mathcal{D}_\alpha$.

Theorem 1.1. *Let $\alpha \in [0,1)$. If f is cyclic in $\mathcal{D}_\alpha$, then*

- *f is an outer function,*
- *$\{\zeta \in \mathbb{T} : f^*(\zeta) = 0\}$ is a set of α-capacity zero.*

The first part is [2, Corollary 1]. For $\alpha = 0$, the second part is [2, Theorem 5], and for general α the proof is similar, the only difference being that the logarithmic kernel k_0 is replaced by k_α. We omit the details.

Our main result is a partial converse to this theorem. To state it, we need to define the notion of a generalized Cantor set.

Let $(a_n)_{n \geq 0}$ be a positive sequence such that $a_0 \leq 2\pi$ and

$$\sup_{n \geq 0} \frac{a_{n+1}}{a_n} < \frac{1}{2}.$$

The *generalized Cantor set* E associated to (a_n) is constructed as follows. Start with a closed arc of length a_0 on the unit circle $\mathbb{T}$. Remove an open arc from the middle, to leave two closed arcs each of length a_1. Then remove two open arcs from their middles to leave four closed arcs each of length a_2. After n steps, we obtain E_n, the union of 2^n closed arcs each of length a_n. Finally, the generalized Cantor set is $E := \cap_n E_n$.

Theorem 1.2. *Let $\alpha \in [0,1)$ and let $f \in \mathcal{D}_\alpha$. Suppose that:*

- *f is an outer function,*
- *$|f|$ extends continuously to $\overline{\mathbb{D}}$,*
- *$\{\zeta \in \mathbb{T} : |f(\zeta)| = 0\}$ is contained in a generalized Cantor set E of α-capacity zero.*

Then f is cyclic for $\mathcal{D}_\alpha$.

Functions f satisfying the hypotheses exist in abundance. Indeed, any generalized Cantor set is a so-called Carleson set, and is thus the zero set of some outer function f such that f and all its derivatives extend continuously to $\overline{\mathbb{D}}$. Moreover, it is very easy to determine which generalized Cantor sets have α-capacity zero. More details will be given in §4.

To prove Theorem 1.2, we adopt the following strategy. In §2, using a technique due to Korenblum, we show that $[f]_{\mathcal{D}_\alpha}$ contains at least those functions $g \in \mathcal{D}_\alpha$ satisfying $|g(z)| \le \operatorname{dist}(z,E)^4$. The idea is then to take one simple such g, and gradually transform it into the constant function 1 while staying inside $[f]_{\mathcal{D}_\alpha}$, thereby proving that $1 \in [f]_{\mathcal{D}_\alpha}$. This requires three tools: a general estimate for weighted Dirichlet integrals of outer functions, some properties of generalized Cantor sets, and a regularization theorem. These tools are developed in §§3,4,5 respectively, and all the pieces are finally assembled in §6, to complete the proof of Theorem 1.2.

Theorem 1.2 was established for the classical Dirichlet space, $\alpha = 0$, in [7, Corollary 1.2]. The proof there followed the same general strategy, but in several places key use was made of a formula of Carleson [4] expressing the Dirichlet integral of an outer function f in terms of the values of $|f^*|$ on the unit circle. No analogue of Carleson's formula is known in the case $0 < \alpha < 1$, and one of the main points of this note is to show how this difficulty may be overcome.

Throughout the paper, we use the notation $C(x_1, \dots, x_n)$ to denote a constant that depends only on $x_1, \dots, x_n$, where the x_j may be numbers, functions or sets. The constant may change from one line to the next.

2. Korenblum's method

Our aim in this section is to prove the following theorem.

Theorem 2.1. *Let $f \in \mathcal{D}_\alpha$ be an outer function such that $|f|$ extends continuously to $\overline{\mathbb{D}}$, and let $F := \{\zeta \in \mathbb{T} : |f(\zeta)| = 0\}$. If $g \in \mathcal{D}_\alpha$ and*

$$|g(z)| \le \operatorname{dist}(z,F)^4 \quad (z \in \mathbb{D}),$$

then $g \in [f]_{\mathcal{D}_\alpha}$.

This theorem is a $\mathcal{D}_\alpha$-analogue of [7, Theorem 3.1], which was proved using a technique of Korenblum. We shall use the same basic technique here. However, the proof in [7] proceeded via a so-called fusion lemma, which, being based on Carleson's formula for the Dirichlet integral, is no longer available to us here. Its place is taken by Corollary 2.3 below.

We need to introduce some notation. Given an outer function f and a Borel subset Γ of $\mathbb{T}$, we define

$$f_\Gamma(z) := \exp\Big(\frac{1}{2\pi}\int_\Gamma \frac{\zeta+z}{\zeta-z}\log|f^*(\zeta)|\,|d\zeta|\Big) \quad (z\in\mathbb{D}).$$

We write $\partial\Gamma$ and Γ^c for the boundary and complement of Γ in $\mathbb{T}$ respectively.

Lemma 2.2. *Let f be a bounded outer function. For every Borel set $\Gamma\subset\mathbb{T}$,*

$$|f_\Gamma'(z)| \le C(f)(|f'(z)| + \operatorname{dist}(z,\partial\Gamma)^{-4}) \quad (z\in\mathbb{D}).$$

Proof. Without loss of generality, we may suppose that $\|f\|_\infty\le 1$. Note that then $\|f_\Gamma\|_\infty \le 1$ for all Γ. Also, obviously, $\log|f^*|\le 0$ a.e. on $\mathbb{T}$, which will help simplify some of the calculations below.

We begin by observing that

$$\frac{f_\Gamma'(z)}{f_\Gamma(z)} = \frac{1}{2\pi}\int_\Gamma \frac{2\zeta}{(\zeta-z)^2}\log|f^*(\zeta)|\,|d\zeta| \quad (z\in\mathbb{D}),$$

from which it follows easily that

$$|f_\Gamma'(z)| \le \frac{2\log(1/|f(0)|)}{\operatorname{dist}(z,\Gamma)^2} \quad (z\in\mathbb{D}). \tag{1}$$

Our aim now is to prove a similar inequality, but with $\partial\Gamma$ in place of Γ. Set $G := \{z\in\mathbb{D} : \operatorname{dist}(z,\Gamma)\ge\operatorname{dist}(z,\Gamma^c)^2\}$. Clearly $\operatorname{dist}(z,\Gamma)\ge\operatorname{dist}(z,\partial\Gamma)^2$ for all $z\in G$, so (1) implies

$$|f_\Gamma'(z)| \le \frac{2\log(1/|f(0)|)}{\operatorname{dist}(z,\partial\Gamma)^4} \quad (z\in G). \tag{2}$$

Now suppose that $z\in\mathbb{D}\setminus G$. Then $\operatorname{dist}(z,\Gamma^c)^2 > \operatorname{dist}(z,\Gamma)\ge(1-|z|^2)/2$, and hence

$$|f_{\Gamma^c}(z)| = \exp\Big(\frac{1}{2\pi}\int_{\Gamma^c}\frac{1-|z|^2}{|\zeta-z|^2}\log|f^*(\zeta)|\,|d\zeta|\Big) \ge |f(0)|^2.$$

Since obviously $f_\Gamma = f/f_{\Gamma^c}$, it follows that, for all $z\in\mathbb{D}\setminus G$,

$$|f_\Gamma'(z)| \le \frac{|f'(z)|}{|f_{\Gamma^c}(z)|} + \frac{|f(z)|}{|f_{\Gamma^c}(z)|^2}|f_{\Gamma^c}'(z)| \le \frac{|f'(z)|}{|f(0)|^2} + \frac{1}{|f(0)|^4}\frac{2\log(1/|f(0)|)}{\operatorname{dist}(z,\Gamma^c)^2},$$

where once again we have used (1), this time with Γ replaced by Γ^c. Noting that $\operatorname{dist}(z,\Gamma^c)\ge\operatorname{dist}(z,\partial\Gamma)$ for all $z\in\mathbb{D}\setminus G$, we deduce that

$$|f_\Gamma'(z)| \le \frac{|f'(z)|}{|f(0)|^2} + \frac{1}{|f(0)|^4}\frac{2\log(1/|f(0)|)}{\operatorname{dist}(z,\partial\Gamma)^2} \quad (z\in\mathbb{D}\setminus G). \tag{3}$$

The inequalities (2) and (3) between them give the result. $\square$

Corollary 2.3. *Let $\alpha\in[0,1)$ and $f\in\mathcal{D}_\alpha\cap H^\infty$ be an outer function. Then, for every Borel set $\Gamma\subset\mathbb{T}$ and every $g\in\mathcal{D}_\alpha$ satisfying $|g(z)|\le\operatorname{dist}(z,\partial\Gamma)^4$, we have*

$$\|f_\Gamma g\|_\alpha \le C(\alpha,f)(1+\|g\|_\alpha).$$

Proof. Using Lemma 2.2, we have

$$|(f_\Gamma g)'| \le |f_\Gamma'||g| + |f_\Gamma||g'| \le C(f)(|f'| + 1 + |g'|).$$

The conclusion follows easily from this. □

Proof of Theorem 2.1. Let I be a connected component of $\mathbb{T} \setminus F$, say $I = (e^{ia}, e^{ib})$. Let $\rho > 1$, and define

$$\psi_\rho(z) := (z-1)^4/(z-\rho)^4,$$
$$\phi_\rho(z) := \psi_\rho(e^{-ia}z)\psi_\rho(e^{-ib}z).$$

The first step is to show that $\phi_\rho f_{\mathbb{T}\setminus I} \in [f]_{\mathcal{D}_\alpha}$.

Let $\epsilon > 0$ and set $I_\epsilon := (e^{i(a+\epsilon)}, e^{i(b-\epsilon)})$ and

$$\phi_{\rho,\epsilon}(z) := \psi_\rho(e^{-i(a+\epsilon)}z)\psi_\rho(e^{-i(b-\epsilon)}z).$$

By Corollary 2.3,

$$\|\phi_{\rho,\epsilon} f_{\mathbb{T}\setminus I_\epsilon}\|_\alpha \le C(f)\|\phi_{\rho,\epsilon}\|_\alpha \le C(f,\rho). \tag{4}$$

Note that $|f_{\mathbb{T}\setminus I_\epsilon}| = |f|$ in a neighborhood of $\mathbb{T} \setminus I$. Since $|f|$ does not vanish inside I, it follows that $|\phi_{\rho,\epsilon} f_{\mathbb{T}\setminus I_\epsilon}|/|f|$ is bounded on $\mathbb{T}$. Also f is an outer function. Therefore, by a theorem of Aleman [1, Lemma 3.1],

$$\phi_{\rho,\epsilon} f_{\mathbb{T}\setminus I_\epsilon} \in [f]_{\mathcal{D}_\alpha}.$$

Using (4), we see that $\phi_{\rho,\epsilon} f_{\mathbb{T}\setminus I_\epsilon}$ converges weakly in $\mathcal{D}_\alpha$ to $\phi_\rho f_{\mathbb{T}\setminus I}$ as $\epsilon \to 0$. Hence $\phi_\rho f_{\mathbb{T}\setminus I} \in [f]_{\mathcal{D}_\alpha}$, as claimed.

Next, we multiply by g. As $g \in \mathcal{D}_\alpha \cap H^\infty$, Aleman's theorem immediately yields $\phi_\rho f_{\mathbb{T}\setminus I} g \in [f]_{\mathcal{D}_\alpha}$. Using the fact that $|g(z)| \le \operatorname{dist}(z,F)^4$, it is easy to check that $\|\phi_\rho g\|_\alpha$ remains bounded as $\rho \to 1$. By Corollary 2.3 again, $\|\phi_\rho f_{\mathbb{T}\setminus I} g\|_\alpha$ is uniformly bounded, and $\phi_\rho f_{\mathbb{T}\setminus I} g$ converges weakly to $f_{\mathbb{T}\setminus I} g$. Hence $f_{\mathbb{T}\setminus I} g \in [f]_{\mathcal{D}_\alpha}$.

Now let $(I_j)_{j\ge1}$ be the complete set of components of $\mathbb{T} \setminus F$, and set $J_n := \cup_1^n I_j$. An argument similar to that above gives $f_{\mathbb{T}\setminus J_n} g \in [f]_{\mathcal{D}_\alpha}$ for all n. Moreover, $\|f_{\mathbb{T}\setminus J_n} g\|_\alpha$ is uniformly bounded. Thus $f_{\mathbb{T}\setminus J_n} g$ converges weakly to g, and so finally $g \in [f]_{\mathcal{D}_\alpha}$. □

3. Estimates for weighted Dirichlet integrals

The following result will act as a partial substitute for Carleson's formula.

Theorem 3.1. *Let $\alpha \in [0,1)$, and let $h : \mathbb{T} \to \mathbb{R}$ be a positive measurable function such that, for every arc $I \subset \mathbb{T}$,*

$$\frac{1}{|I|}\int_I h(\zeta)\,|d\zeta| \ge |I|^\alpha. \tag{5}$$

If f is an outer function, then

$$\mathcal{D}_\alpha(f) \le \frac{1}{\pi}\iint_{\mathbb{T}^2} \frac{(|f^*(\zeta)|^2 - |f^*(\zeta')|^2)(\log|f^*(\zeta)| - \log|f^*(\zeta')|)}{|\zeta-\zeta'|^2}(h(\zeta)+h(\zeta'))|d\zeta||d\zeta'|.$$

Proof. Given $z \in \mathbb{D}$, let I_z be the arc of $\mathbb{T}$ with midpoint $z/|z|$ and arclength $|I_z| = 2(1-|z|^2)$. For $\zeta \in I_z$, we have $|\zeta - z| \le 2(1-|z|^2)$, and so

$$P(z,\zeta) := \frac{1-|z|^2}{|\zeta - z|^2} \ge \frac{1}{4(1-|z|^2)} = \frac{1}{2|I_z|}.$$

Hence, using (5), we have

$$\int_{\mathbb{T}} P(z,\zeta)h(\zeta)\,|d\zeta| \ge \frac{1}{2|I_z|}\int_{I_z} h(\zeta)\,|d\zeta| \ge \frac{1}{2}|I_z|^\alpha \ge \frac{1}{2}(1-|z|^2)^\alpha.$$

Therefore, by Fubini's theorem,

$$\mathcal{D}_\alpha(f) := \frac{1}{\pi}\int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^\alpha\,dA(z) \le 2\int_{\mathbb{T}} \mathcal{D}_\zeta(f)h(\zeta)\,|d\zeta|,$$

where

$$\mathcal{D}_\zeta(f) := \frac{1}{\pi}\int_{\mathbb{D}} |f'(z)|^2 P(z,\zeta)\,dA(z) \quad (\zeta \in \mathbb{T}).$$

Now $\mathcal{D}_\zeta(f)$ is the so-called local Dirichlet of integral of f at ζ, which was studied in detail by Richter and Sundberg in [9]. In particular, they showed that, if f is an outer function, then

$$\mathcal{D}_\zeta(f) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{|f^*(\zeta)|^2 - |f^*(\zeta')|^2 - 2|f^*(\zeta')|^2 \log|f^*(\zeta)/f^*(\zeta')|}{|\zeta - \zeta'|^2}\,|d\zeta'|.$$

Substituting this into the preceding estimate for $\mathcal{D}_\alpha(f)$, and noting the obvious fact that $h(\zeta) \le h(\zeta) + h(\zeta')$, we deduce that $\mathcal{D}_\alpha(f)$ is majorized by

$$\frac{1}{\pi}\iint_{\mathbb{T}} \frac{|f^*(\zeta)|^2 - |f^*(\zeta')|^2 - 2|f^*(\zeta')|^2 \log|f^*(\zeta)/f^*(\zeta')|}{|\zeta - \zeta'|^2}(h(\zeta)+h(\zeta'))\,|d\zeta'|\,|d\zeta|.$$

Exchanging the roles of ζ and ζ', we see that $\mathcal{D}_\alpha(f)$ is likewise majorized by

$$\frac{1}{\pi}\iint_{\mathbb{T}} \frac{|f^*(\zeta')|^2 - |f^*(\zeta)|^2 - 2|f^*(\zeta)|^2 \log|f^*(\zeta')/f^*(\zeta)|}{|\zeta' - \zeta|^2}(h(\zeta')+h(\zeta))\,|d\zeta|\,|d\zeta'|.$$

Taking the average of these last two estimates, we obtain the inequality in the statement of the theorem. □

We are going to apply this result with $h(\zeta) := Cd(\zeta,E)^\alpha$, where C is a constant, d denotes arclength distance on $\mathbb{T}$, and E is a closed subset of $\mathbb{T}$. Condition (5) thus becomes

$$\frac{1}{|I|}\int_I d(\zeta,E)^\alpha\,|d\zeta| \ge C^{-1}|I|^\alpha \quad \text{for all arcs } I \subset \mathbb{T}. \tag{6}$$

A set E which satisfies this condition for some α, C is called a *K-set* (after Kotochigov). K-sets arise as the interpolation sets for certain function spaces, and have several other interesting properties. We refer to [3, §1] and [6, §3] for more details. In particular, if E satisfies (6), then it has measure zero and $\log d(\zeta,E) \in L^1(\mathbb{T})$.

Theorem 3.2. *Let $\alpha \in (0,1)$, let E be a closed subset of $\mathbb{T}$ satisfying (6), and let $w : [0,2\pi] \to \mathbb{R}^+$ be an increasing function such that $t \mapsto \omega(t^\gamma)$ is concave for some $\gamma > 2/(1-\alpha)$. Let f_w be the outer function satisfying*

$$|f_w^*(\zeta)| = w(d(\zeta,E)) \qquad a.e \text{ on } \mathbb{T}.$$

Then

$$\mathcal{D}_\alpha(f_w) \le C(\alpha,\gamma,E) \int_{\mathbb{T}} w'(d(\zeta,E))^2 d(\zeta,E)^{1+\alpha}\,|d\zeta|. \tag{7}$$

In particular $f_w \in \mathcal{D}_\alpha$ if the last integral is finite.

Proof. The proof is largely similar to that of [7, Theorem 4.1], so we give just a sketch, concentrating on those parts where the two proofs differ.

We begin by remarking that the concavity condition on w easily implies that $|\log w(d(\zeta,E))| \le C(w)|\log d(\zeta,E)|$, so $\log w(d(\zeta,E)) \in L^1(\mathbb{T})$ and the definition of f_w makes sense.

By Theorem 3.1, we have

$$\mathcal{D}_\alpha(f_w) \le C(\alpha,E) \iint_{\mathbb{T}^2} \frac{(w^2(\delta)-w^2(\delta'))(\log w(\delta)-\log w(\delta'))}{|\zeta-\zeta'|^2}(\delta^\alpha + \delta'^\alpha)\,|d\zeta|\,|d\zeta'|,$$

where we have written $\delta := d(\zeta,E)$ and $\delta' := d(\zeta',E)$.

Let (I_j) be the connected components of $\mathbb{T}\setminus E$, and set

$$N_E(t) := 2\sum_j 1_{\{|Ij|>2t\}} \quad (0 < t \le \pi).$$

Then, for every measurable function $\Omega : [0,\pi] \to \mathbb{R}^+$, we have

$$\int_{\mathbb{T}} \Omega(d(\zeta,E))\,|d\zeta| = \int_0^\pi \Omega(t) N_E(t)\,dt.$$

In particular, as in [7], it follows that

$$\begin{aligned}&\iint_{\mathbb{T}^2} \frac{(w^2(\delta)-w^2(\delta'))(\log w(\delta)-\log w(\delta'))}{|\zeta-\zeta'|^2}(\delta^\alpha + \delta'^\alpha)\,|d\zeta|\,|d\zeta'| \\ &\le C(\alpha)\int_0^\pi\int_0^\pi \frac{(w^2(s+t)-w^2(t))(\log w(s+t)-\log w(t))}{s^2}(s+t)^\alpha N_E(t)\,ds\,dt.\end{aligned}$$

The concavity assumption on w implies that $t \to t^{1-1/\gamma}w'(t)$ is decreasing, and thus, as in [7],

$$\begin{aligned} w^2(t+s)-w^2(t) &\le 2\gamma w(t+s)w'(t)t\big((1+s/t)^{1/\gamma}-1\big), \\ \log w(t+s)-\log w(t) &\le tw'(t)\frac{(1+s/t)^{1/\gamma}}{w(t+s)}\log(1+s/t). \end{aligned}$$

Combining these estimates, we obtain

$$
\begin{aligned}
&\int_0^\pi \int_0^\pi \frac{(w^2(t+s)-w^2(t))(\log w(t+s)-\log w(t))}{s^2}(s+t)^\alpha\, ds N_E(t)\, dt \\
&\le \int_0^\pi \int_0^\pi 2\gamma w'(t)^2 t^{2+\alpha}\big((1+s/t)^{1/\gamma}-1\big)(1+s/t)^{1/\gamma+\alpha}\log(1+s/t)\,\frac{ds}{s^2}N_E(t)\, dt \\
&= \int_0^\pi 2\gamma w'(t)^2 t^{1+\alpha}\Big(\int_0^{\pi/t} 2\gamma\big((1+x)^{1/\gamma}-1\big)(1+x)^{1/\gamma+\alpha}\log(1+x)\frac{dx}{x^2}\Big)N_E(t)\, dt \\
&\le C(\alpha,\gamma)\int_0^\pi w'(t)^2 t^{1+\alpha} N_E(t)\, dt.
\end{aligned}
$$

In the last inequality we used the fact that $\gamma > 2/(1-\alpha)$. □

4. Generalized Cantor sets

The notion of the generalized Cantor set E associated to a sequence (a_n) was defined in §1. In this section we briefly describe some pertinent properties of these sets. We shall write

$$\lambda_E := \sup_{n\ge 0} \frac{a_{n+1}}{a_n}.$$

Recall that, by hypothesis, $\lambda_E < 1/2$.

Our first result shows that generalized Cantor sets satisfy (5), and hence that Theorem 3.2 is applicable to such sets.

Proposition 4.1. *Let E be a generalized Cantor set and let $\alpha \in [0,1)$. Then, for each arc $I \subset \mathbb{T}$,*

$$\frac{1}{|I|}\int_I d(\zeta,E)^\alpha\,|d\zeta| \ge C(\alpha,\lambda_E)|I|^\alpha.$$

Proof. Let I be an arc with $|I| \le 2a_0$, and choose n so that $2a_n < |I| \le 2a_{n-1}$. Recall that the n-th approximation to E consists of 2^n arcs, each of length a_n, and that the distance between these arcs is at least $a_{n-1}-2a_n$. If I meets at least two of these arcs, then $I\setminus E$ contains an arc J of length $a_{n-1}-2a_n$, and if I meets at most one of these arcs, then $I\setminus E$ contains an arc J of length $(|I|-a_n)/2$. Thus $I\setminus E$ always contains an arc J such that $|J|/|I| \ge \min\{1/2-\lambda_E,\, 1/4\}$. Consequently

$$\frac{1}{|I|}\int_I d(\zeta,E)^\alpha\,|d\zeta| \ge \frac{1}{|I|}\int_J d(\zeta,E)^\alpha\,|d\zeta| \ge \frac{1}{|I|}\frac{|J|^{\alpha+1}}{\alpha+1} \ge C(\alpha,\lambda_E)|I|^\alpha.$$

□

In the next two results, we write $E_t := \{\zeta\in\mathbb{T} : d(\zeta,E)\le t\}$.

Proposition 4.2. *If E is a generalized Cantor set, then $|E_t| = O(t^\mu)$ as $t\to 0$, where $\mu := 1-\log 2/\log(1/\lambda_E)$.*

Proof. Given $t \in (0, a_0]$, choose n so that $a_n < t \le a_{n-1}$. Then clearly $|E_t| \le 2^n(a_n+2t) \le 3.2^n t$. Also $2^{n-1} = (1/\lambda_E^{n-1})^{\log 2/\log(1/\lambda_E)} \le (a_0/t)^{\log 2/\log(1/\lambda_E)}$. Hence $|E_t| \le C(a_0, \lambda_E) t^{1-\log 2/\log(1/\lambda_E)}$. □

In particular, every generalized Cantor set E is a *Carleson set*, that is, $\int_0^\pi (|E_t|/t)\, dt < \infty$. Taylor and Williams [10] showed that Carleson sets are zero sets of outer functions in $A^\infty(\mathbb{D})$. This justifies a remark made in §1.

The final property that we need concerns the α-capacity, C_α, which was defined in §1.

Theorem 4.3. *Let E be a generalized Cantor set and let $\alpha \in [0, 1)$. Then*

$$C_\alpha(E) = 0 \iff \int_0^\pi \frac{dt}{t^\alpha |E_t|} = \infty.$$

Proof. This follows easily from [5, §IV, Theorems 2 and 3]. □

5. Regularization

We shall need the following regularization result. The proof is the same, with minor modifications, as that of [7, Theorem 5.1].

Theorem 5.1. *Let $\alpha \in [0, 1)$, let $\sigma \in (0, 1)$ and let $a > 0$. Let $\phi : (0, a] \to \mathbb{R}^+$ be a function such that*

- *$\phi(t)/t$ is decreasing,*
- *$0 < \phi(t) \le t^\sigma$ for all $t \in (0, a]$,*
- $\displaystyle\int_0^a \frac{dt}{t^\alpha \phi(t)} = \infty.$

Then, given $\rho \in (0, \sigma)$, there exists a function $\psi : (0, a] \to \mathbb{R}^+$ such that

- *$\psi(t)/t^\rho$ is increasing,*
- *$\phi(t) \le \psi(t) \le t^\sigma$ for all $t \in (0, a]$,*
- $\displaystyle\int_0^a \frac{dt}{t^\alpha \psi(t)} = \infty.$

6. Completion of the proof of Theorem 1.2

For $\alpha = 0$, this theorem was proved in [7, §6]. The proof for $\alpha \in (0, 1)$ will follow the same general lines, and once again we shall concentrate mainly on the places where the proofs differ.

Let f be the function in the theorem. Our goal is to show that $1 \in [f]_{\mathcal{D}_\alpha}$.

Let E be as in the theorem, and let g be the outer function such that

$$|g^*(\zeta)| = d(\zeta, E)^4 \quad \text{a.e. on } \mathbb{T}.$$

Then $|g(z)| \le (\pi/2)^4 \operatorname{dist}(z, E)^4$ $(z \in \mathbb{D})$: indeed, for every $\zeta_0 \in E$, we have

$$\log |g(z)| \le \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1 - |z|^2}{|z - \zeta|^2} 4 \log\bigl((\pi/2)|\zeta - \zeta_0|\bigr)\, |d\zeta| = 4 \log\bigl((\pi/2)|z - \zeta_0|\bigr).$$

By assumption, the zero set $F := \{\zeta \in \mathbb{T} : |f(\zeta)| = 0\}$ in contained in E, so $|g(z)| \le (\pi/2)^4 \operatorname{dist}(z,F)^4$ $(z \in \mathbb{D})$. Theorem 2.1 therefore applies, and we can infer that $g \in [f]_{\mathcal{D}_\alpha}$. It thus suffices to prove that $1 \in [g]_{\mathcal{D}_\alpha}$.

We shall construct a family of functions $w_\delta : [0,\pi] \to \mathbb{R}^+$ for $0 < \delta < 1$ such that the associated outer functions f_{w_δ} belong to $[g]_{\mathcal{D}_\alpha}$ and satisfy:

(i) $|f_{w_\delta}^*| \to 1$ a.e. on $\mathbb{T}$ as $\delta \to 0$,
(ii) $|f_{w_\delta}(0)| \to 1$ as $\delta \to 0$,
(iii) $\liminf_{\delta\to 0} \|f_{w_\delta}\|_\alpha < \infty$.

If such a family exists, then a subsequence of the f_{w_δ} converges weakly to 1 in $\mathcal{D}_\alpha$, and since they all belong to $[g]_{\mathcal{D}_\alpha}$, it follows that $1 \in [g]_{\mathcal{D}_\alpha}$, as desired.

By Proposition 4.2, there exists $\mu > 0$ such that $|E_t| = O(t^\mu)$ as $t \to 0$. Fix ρ, σ satisfying

$$\frac{1-\alpha}{2} < \rho < \sigma < \min\Big\{1-\alpha,\ \frac{1-\alpha+\mu}{2}\Big\}.$$

Define $\phi : (0,\pi] \to \mathbb{R}^+$ by

$$\phi(t) := \max\Big\{\min\{|E_t|,\ t^\sigma\},\ t^{1-\alpha}\Big\} \quad (t \in (0,\pi]).$$

Clearly $\phi(t)/t$ is increasing and $0 \le \phi(t) \le t^\sigma$ for all t. We claim also that

$$\int_0^\pi \frac{dt}{t^\alpha \phi(t)} = \infty. \tag{8}$$

To see this, note that

$$\int_t^\pi \frac{ds}{s^\alpha |E_s|} \ge \frac{t}{|E_t|}\int_t^\pi \frac{ds}{s^{\alpha+1}} = C(\alpha)\frac{t^{1-\alpha}}{|E_t|},$$

whence

$$\begin{aligned}\int_\epsilon^\pi \frac{dt}{t^\alpha\phi(t)} &\ge \int_\epsilon^\pi \frac{dt}{\max\{t, t^\alpha|E_t|\}}\\ &\ge C(\alpha)\int_\epsilon^\pi \frac{ds}{t^\alpha|E_t|(\int_t^\pi ds/s^\alpha|E_s|)}\\ &\ge C(\alpha)\log\int_\epsilon^\pi \frac{dt}{t^\alpha|E_t|}.\end{aligned}$$

Since E is a generalized Cantor set of α-capacity zero, Theorem 4.3 shows that

$$\int_0^\pi \frac{dt}{t^\alpha|E_t|} = \infty.$$

Consequently (8) holds, as claimed.

We have now shown that ϕ satisfies all the hypotheses of the regularization theorem, Theorem 5.1. Therefore there exists a function $\psi : (0,\pi] \to \mathbb{R}^+$ satisfying the conclusions of that theorem, namely: $\psi(t)/t^\rho$ is increasing, $t^{1-\alpha} \le \phi(t) \le \psi(t) \le t^\sigma$ for all t, and $\int_0^\pi dt/t^\alpha\psi(t) = \infty$.

For $0 < \delta < 1$, we define $w_\delta : [0, \pi] \to \mathbb{R}^+$ by

$$w_\delta(t) := \begin{cases} \dfrac{\delta^\rho}{\psi(\delta)} t^{1-\alpha-\rho} & 0 \le t \le \delta \\ A_\delta - \log \displaystyle\int_t^\pi \frac{ds}{s^\alpha \psi(s)} & \delta < t \le \eta_\delta \\ 1 & \eta_\delta < t \le \pi, \end{cases}$$

where A_δ and η_δ are constants chosen to make w_δ continuous.

Let us show that $f_{w_\delta} \in [g]_{\mathcal{D}_\alpha}$. Note first that $w_\delta(t)/t^{1-\alpha-\rho}$ is a bounded function. Therefore $f_{w_\delta}/g^{(1-\alpha-\rho)/4}$ is bounded. Using Theorem 3.2, we have $g^{(1-\alpha-\rho)/4} \in \mathcal{D}_\alpha$. Consequently, by a theorem of Aleman [1, Lemma 3.1], $f_{w_\delta} \in [g^{(1-\alpha-\rho)/4}]_{\mathcal{D}_\alpha}$. Using another result of Aleman [1, Theorem 2.1], we have $g^{(1-\alpha-\rho)/4} \in [g]_{\mathcal{D}_\alpha}$. Hence $f_{w_\delta} \in [g]_{\mathcal{D}_\alpha}$, as claimed.

It remains to check that the functions f_{w_δ} satisfy properties (i)–(iii) above. The verifications run along the same lines as those in [7, §6], using the properties of ψ above, and Theorem 3.2 in place of [7, Theorem 4.1].

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